Teleportation from a Projection Operator Point of View

N. Erez[1]

[1]*Institute for Quantum Studies and Physics Dept.,*
Texas A&M University, College Station, TX 77843-4242

The process of quantum state teleportation is described from the point of view of the properties of projections onto one-dimensional subspaces. It is introduced as a generalization of the remote preparation of a known state by use of an EPR pair. The discrete and continuous cases are treated in a unified way. The conceptual and calculational simplicity is pedagogically advantageous.

I. INTRODUCTION

Quantum state teleportation has become, since its introduction a decade ago by Bennett et al.[5], one of the most important (conceptual) applications of entanglement. This is illustrated by the interesting fact that almost every recent paper on entanglement begins by citing the use of that resource for teleportation. Although the main result is given by a single equation which can be verified with a few lines of elementary algebra (Equation 5 of [5]), it is subtle and raises many questions: What is the precise role of entanglement, and how does it relate to other manifestations of it? What role do the $(SU(2))$ group properties play? That there was more than meets the eye to this apparently irreducible derivation, was strikingly illustrated by Vaidman in his interpretation of teleportation [7] in terms of a class of measurements introduced much earlier by Aharonov and Albert[3], which suggested to him a generalization to the teleportation of states of a particle with a continuous degree of freedom.

To me, teleportation is still easier calculated than understood. I think the most natural motivation is the remote preparation of a known spin state[6] using a measurement on the other half of an EPR-Bohm pair, which works half the time (and registers when it fails). State teleportation can then be viewed as a generalization to the remote preparation of an unknown state determined by the state of an auxiliary particle. Interestingly, this problem is in some respects simpler, and an efficient corrective step can be implemented when the direct process fails (3/4 of the time in this case). In view of the linearity of projection operators, the main trick is to find the appropriate basis corresponding to the (two-particle) measurements. The key is the basis element for which no correction is needed. Continuous teleportation is then described in analogous fashion.

The main results require nothing more than elementary linear algebra (as is also true of the original teleportation paper). A few simple group theoretic and information theoretic results are mentioned, but only for motivation, or in comments.

II. EPR-BOHM AND REMOTE STATE PREPARATION

The concept of entanglement had its origin in the famous EPR paper[1]. It is often more convenient to use Bohm's version[2] of a spin singlet state of two spatially separated spin $1/2$ particles. That is also our starting point.

The next two paragraphs will introduce remote state preparation[6] using Pauli spin algebra terminology and properties (e.g., $|\uparrow_{\hat{n}}\rangle$ will be used for the state with spin projection $+\hbar/2$ along the $\hat{n}$ axis). This will only serve as heuristic motivation. Once done, however, it will prove profitable to rephrase the result in plain linear algebraic language.

Suppose two experimenters, 'Alice' and 'Bob', share an EPR-Bohm state, i.e., each has a spin $1/2$ particle localized spatially in a small space, but both sharing a total spin state of zero spin: $\left(\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\right)_{12}\psi(x_1-x_{\text{Alice}})\psi(x_2-x_{\text{Bob}})$. At this point, the two particles each have an undetermined spin state. Alice would like Bob's particle to be in the state $|\uparrow_{\hat{n}}\rangle$. She can perform a measurement on her own particle with respect to the $\hat{n}$ axis (i.e., the measurement basis is $\{|\uparrow_{\hat{n}}\rangle, |\downarrow_{\hat{n}}\rangle\}$). The two possible outcomes of the experiment can be expressed as follows:

$$\begin{cases} (|\uparrow_{\hat{n}}\rangle\langle\uparrow_{\hat{n}}|)_1 \left(\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\right)_{12} = \frac{1}{\sqrt{2}}|\uparrow_{\hat{n}}\rangle_1|\downarrow_{\hat{n}}\rangle_2 \\ (|\downarrow_{\hat{n}}\rangle\langle\downarrow_{\hat{n}}|)_1 \left(\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\right)_{12} = \frac{1}{\sqrt{2}}|\downarrow_{\hat{n}}\rangle_1|\uparrow_{\hat{n}}\rangle_2 \end{cases} \tag{1}$$

where we have used the symmetry property that $\frac{|\uparrow_{\hat{n}}\downarrow_{\hat{n}}\rangle-|\downarrow_{\hat{n}}\uparrow_{\hat{n}}\rangle}{\sqrt{2}} = \frac{|\uparrow_{\hat{n}'}\downarrow_{\hat{n}'}\rangle-|\downarrow_{\hat{n}'}\uparrow_{\hat{n}'}\rangle}{\sqrt{2}}$ for all $\hat{n}, \hat{n}'$ (provided we had been careful to define the phases of the $\{|\uparrow_{\hat{n}}\rangle\}$ consistently).

This means that half the time (i.e., with probability 1/2) Alice measures $|\downarrow_{\hat{n}}\rangle_1$ and she can tell Bob his particle has the desired spin state. What about the other half? Alice tells Bob his particle's spin is flipped. Can he fix it? The operation $|\uparrow_{\hat{n}}\rangle \mapsto |\downarrow_{\hat{n}}\rangle$ $\forall \hat{n}$, is not unitary. There are unitary operations that flip a $|\uparrow_{\hat{n}}\rangle$ for a *given* $\hat{n}$, namely rotations about an axis perpendicular to $\hat{n}$. But if Bob knows the direction $\hat{n}$, he might as well prepare $|\uparrow_{\hat{n}}\rangle$ locally.

Now let us rephrase this in simple linear algebraic terms. Fix an orthonormal basis: $\{|\uparrow\rangle, |\downarrow\rangle\}$ where 'up' and 'down' will only be taken as convenient labels[12]. Thus, $|\uparrow_{\hat{n}}\rangle = \alpha^*|\uparrow\rangle + \beta^*|\downarrow\rangle$ and $|\downarrow_{\hat{n}}\rangle = \beta|\uparrow\rangle - \alpha|\downarrow\rangle$ for some $\alpha, \beta \in \mathbb{C}$ (neglecting an inconsequential common phase). We shall not use the symmetry of the singlet state with respect to rotation, and it is convenient to replace this 2-particle state shared by Alice and Bob by $\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$ (in fact, we can just relabel Alice's basis states so as to give the original state this form with respect to the new basis). Equation 1 now reads:

$$\begin{cases} [(\alpha^*|\uparrow\rangle + \beta^*|\downarrow\rangle)(\alpha\langle\uparrow| + \beta\langle\downarrow|)]_1 \left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{12} = \frac{1}{\sqrt{2}}(\alpha^*|\uparrow\rangle + \beta^*|\downarrow\rangle)_1 (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_2 \\ [(\beta|\uparrow\rangle - \alpha|\downarrow\rangle)(\beta^*\langle\uparrow| - \alpha^*\langle\downarrow|)]_1 \left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{12} = \frac{1}{\sqrt{2}}(\beta|\uparrow\rangle - \alpha|\downarrow\rangle)_1 (\beta^*|\uparrow\rangle - \alpha^*|\downarrow\rangle)_2 \end{cases} \quad (2)$$

We interpret this as follows. Bob would like his particle's state to be $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$ where α, β are determined by Alice. She sets her measuring apparatus to measure in the basis $\{\alpha^*|\uparrow\rangle + \beta^*|\downarrow\rangle, \beta|\uparrow\rangle - \alpha|\downarrow\rangle\}$. If she finds her particle to be in the first state (which happens with probability 1/2), she tells Bob that his particle has the desired state. If she doesn't, she tells him it is in the orthogonal state. In the latter case, it is clear from the form of the α, β dependence that Bob's state is related to the desired state by an anti-unitary transformation. We also see that we can understand the preparation, when successful, in terms of the (sesqui-)linearity of the projection operation. It should be noted that in an actual experimental application of this scheme, the spinor picture *is* likely to play a role (e.g., in the settings of a Stern-Gerlach apparatus).

III. QUANTUM STATE TELEPORTATION AND REMOTE STATE PREPARATION

The last observation suggests a generalization. If the parameters α, β are first encoded in the state of an auxiliary particle, we can use a fixed basis for the measurements (now, joint measurements of the auxiliary particle and Alice's particle). Instead of the first line of Equation 2 we write:

$$\left[\frac{1}{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|)\right]_{12} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_1 \left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{23} = \frac{1}{2}\left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{12} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_3 \quad (3)$$

Alice doesn't need to know the values of α, β to make the measurements (the state of particle 1 can be prepared by someone else). A related and very important difference from the previous situation is that if particle 1 is originally in an entangled state with another system, the effect of a measurement with this outcome is to replace particle 1 by particle 3 in the overall state. This can be verified by inspection–in the Dirac bracket notation, all we have to do is let α, β take 'ket' rather than 'c-number' values.

The two-particle state measurement is, of course, not determined by Equation 3. Any measurement basis will determine a set of linear transformations of the state of particle 1 into that of particle 3. However, we would like them to be unitary (since this is equivalent to the requirement that the inverse, corrective, operations be unitary). The remarkable paper by Bennett et al.[5] uses the basis[4]: $\left\{\frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}}, \frac{|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}}\right\}$ (this is known as the Bell basis). Let us write out explicitly the rest of the measurement outcomes[13]:

$$\left[\frac{1}{2}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)(\langle\uparrow\uparrow| - \langle\downarrow\downarrow|)\right]_{12} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_1 \left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{23} = \frac{1}{2}\left(\frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{12} (\alpha|\uparrow\rangle - \beta|\downarrow\rangle)_3 \quad (4)$$

$$\left[\frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)\right]_{12} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_1 \left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{23} = \frac{1}{2}\left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\right)_{12} (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)_3 \quad (5)$$

$$\left[\frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)\right]_{12} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_1 \left(\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\right)_{23} = \frac{1}{2}\left(\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right)_{12} (\alpha|\downarrow\rangle - \beta|\uparrow\rangle)_3 \quad (6)$$

And the correction operations Bob needs to perform in each case, are readily seen to be unitary (e.g., $|\uparrow\rangle \mapsto |\uparrow\rangle$, $|\downarrow\rangle \mapsto -|\downarrow\rangle$ for (4)).

Comment. Let $\{|u_i\rangle_{12}\}_{i=1..4}$ be the basis corresponding to any nondegenerate two particle measurement for our two Pauli spinors. Then as noted by Schrödinger, there exists for $|u_i\rangle$ a so-called Schmidt decomposition:

$$|u_j\rangle = \sum_{i=1..2} \lambda_i^j |a_i^j\rangle_1 |b_i^j\rangle_2 \tag{7}$$

where $\left\{|a_i^j\rangle\right\}_{i=1..2}$ and $\left\{|b_i^j\rangle\right\}_{i=1..2}$ are orthonormal bases of the respective single particle Hilbert spaces. It is clear that for such a basis to induce unitary transformations of the state of particle 1 to that of 3, for all i, j we should have $|\lambda_i^j| = \frac{1}{\sqrt{2}}$[14]. This suggests a direct generalization to the teleportation of states in an arbitrary finite dimensional Hilbert space[15].

Finally, let us note that the Bell basis can be characterized as the simultaneous eigenstates of the operators $\sigma_{z1}\sigma_{z2}$ (spin-z alignment) and $\sigma_{x1}\sigma_{x2}$ (relative phase of the two terms). The correction operators also have elegant expressions in the Pauli algebra $(1, \sigma_z, \sigma_x, \sigma_z\sigma_x$; respectively).

IV. CONTINUOUS TELEPORTATION

Consider now the teleportation of a particle with a one-dimensional continuous degree of freedom. It is straightforward to write down the formal analog of Equation 3:

$$\left(\int dx dx' |x, x\rangle\langle x', x'|\right)_{1,2} \int dx'' \psi(x'') |x''\rangle_1 \int dx''' |x''', x'''\rangle_{2,3} = \int dx |x, x\rangle_{1,2} \int dx' \psi(x') |x'\rangle_3 \tag{8}$$

Thus, if Alice measures the state of the particles 1 and 2 in a basis that includes the element $\int dx |x, x\rangle$ and is fortunate enough to find them in this particular state, she has successfully 'teleported' the state of particle 1 to particle 3. We now need analogs for Equation 4-6, i.e., to extend $\int dx |x, x\rangle$ to an appropriate basis of the two-particle Hilbert space. We have considerable freedom. The analogy suggests the following general form:

$$\int dx \exp(i\phi(x)) |x, x'(x)\rangle \tag{9}$$

If we require that $x'(x)$ be smooth, the comment on the Schmidt coefficients for the discrete case suggests that $\left|\frac{\partial x'}{\partial x}\right| = 1$, and so $x'(x)$ should have the form $x'(x) = \pm x + \alpha$. The $\phi(x)$ should be chosen so as to ensure completeness. The problem is to find a basis that corresponds to a natural measurement.

Following the original continuous teleportation paper by Vaidman [7], we note that the Bell basis, used in the discrete case, can also be characterized as the eigenbasis of the operators $(\sigma_{y1} + \sigma_{y2}) \, mod \, 4$ and $(\sigma_{x1} + \sigma_{x2}) \, mod \, 4$ (which are linear in the operators). These operators had been investigated much earlier by Aharonov and Albert [3], in the context of non-local measurements. The analogous operators for the one-dimensional continuous case were $x_1 + x_2, p_1 - p_2$ (as in the original EPR paper). This led him to suggest both a new interpretation of quantum teleportation, and a generalization to the continuous case. From the present point of view, it suggests the eigenbasis of of the operators $x_1 - x_2, p_1 + p_2$, namely, $\left\{\int dx \exp(-i\beta x) |x, x + \alpha\rangle\right\}_{\alpha, \beta \in \mathbb{R}}$(where it has been tacitly assumed that x values are given in terms of some given unit making the expressions dimensionless). The analogs of Eqns. 4-6 are then:

$$\left(\int dx dx' e^{i\beta(x-x')} |x, x + \alpha\rangle\langle x', x' + \alpha|\right)_{1,2} \int dx'' \psi(x'') |x''\rangle_1 \int dx''' |x''', x'''\rangle_{2,3} =$$

$$\int dx e^{i\beta x} |x, x + \alpha\rangle_{1,2} \int dx' \psi(x') e^{-i\beta x'} |x' + \alpha\rangle_3 \quad \alpha, \beta \in \mathbb{R} \tag{10}$$

And the (unitary) correction operations Bob needs to take when Alice measures $x_1 - x_2 = -\alpha$, $p_1 + p_2 = -\beta$ correspond to the operations: $x_3 \mapsto x_3 - \alpha$, $p_3 \mapsto p_3 - \beta$.

I shall not attempt to justify the formal formulas of this section in terms of measure theory. That would defeat the aim to keep this introduction elementary, but more importantly, the limiting case of delta-function correlations

considered here is not only highly singular, but also physically unrealistic. The more important question of realistic implementations was treated in the beautiful paper of Braunstein and Kimble[8] which suggested a realizable experiment in the realm of quantum optics. An elementary account of their scheme is given in [11].

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[2] D. Bohm, *Quantum Theory* (Prentice Hall, Englewood Cliffs, NJ, 1951).

[3] Y. Aharonov and D. Albert, Phys. Rev. D **21**, 3316 (1980); **24**, 359 (1981).

[4] S.L. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. **68**, 3259 (1992).

[5] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[6] A.K. Pati, quant-ph/9907022; Phys. Rev. A **63**, 014320 (2001).

[7] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).

[8] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[9] A. Furusawa, J.L. Sorensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, and E.S. Polzik, *Science* , **282**, no.5389, 706 (1998),

[10] N. Takei, H. Yonezawa, T. Aoki, and A. Furusawa, Phys. Rev. Lett. **94**, 220502 (2005).

[11] L. Vaidman, N. Erez, and A. Retzker, quant-ph/0507051, to be published in Int. J. Quan. Inf.

[12] The notation is justified by the fact that there exists an axis with respect to which this corresponds to the 'up' and 'down' states, for appropriate choice of phases. We do not need this property.

[13] Note that the sum of equations 3-6 is $(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_1 |\psi^+\rangle_{23} = |\psi^+\rangle_{12} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)_3 + |\psi^-\rangle_{12} (\alpha|\uparrow\rangle - \beta|\downarrow\rangle)_3 + |\phi^+\rangle_{12} (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)_3 + |\phi^-\rangle_{12} (\alpha|\downarrow\rangle - \beta|\uparrow\rangle)_3$. This just Eq. 5 of [5], adapted to the case where the "quantum channel" consists of a pair of spin-1/2 particles in a ψ^+, rather than ϕ^- state.

[14] This is also the condition for the $|u_i\rangle$ to be maximally entangled states.

[15] In some cases it might be advantageous to imbed the Hilbert space, by using ancillary particles if necessary, into one with dimensionality 2^k, for some k, and teleport the state one 2-dimensional degree of freedom ('qubit') at a time. This is possible by the remark below Equation 3.